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Average Widths and Optimal Recovery of Multivariate Besov Classes in $L_p(R^d)$

Jiang Yanjie

Department of Foundation, North China Electric Power University, Baoding 071003, People's Republic of China

and

Liu Yongping[1]

Department of Mathematics, Beijing Normal University, Beijing 100875, People's Republic of China
E-mail: ypliu@bnu.edu.cn

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In this paper, we obtain some weak asymptotic results for average Kolmogorov σ-width and average linear σ-width of some isotropic and anisotropic Besov classes. The asymptotic behavior of optimal recovery is established for isotropic Besov classes. © 2000 Academic Press

Key Words: average dimension; average width; optimal recovery; Besov class.

1. INTRODUCTION

1.1. *Average Widths and Optimal Recovery*

Let $1 \leqslant p \leqslant \infty$, let E be a measurable subset of R^d, and let $L_p(E)$ denote the Banach space of measurable functions $x(\cdot)$ on E with the norm

$$\|f\|_{L_p(E)} := \left\{ \int_E |f(x)|^p \, dx \right\}^{1/p}, \qquad 1 \leqslant p < \infty,$$

$$\|f\|_{L_\infty(E)} := \operatorname*{ess\,sup}_{x \in E} |f(x)|, \qquad p = \infty.$$

For convenience, we write $\|\cdot\|_p$ instead of $\|\cdot\|_{L_p(R^d)}$.

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In [2, 3, 5, 7], the authors studied some problems of the average widths and optimal recovery of Sobolev classes of smooth functions defined on the Euclidean space R^d in the metric of $L_q(R^d)$ and obtained some exact results (for $d=1$) and some weak asymptotic results (for $d>1$), respectively. In the present paper, we study the same problems in Besov spaces defined on R^d. To do these, as in [2, 5], we mention some needed terminology here.

Let $\alpha>0$ and let P_α be the continuous linear operator in $L_p(R^d)$ defined by $P_\alpha := \chi_\alpha(\cdot)\, x(\cdot)$, where $\chi_\alpha(\cdot)$ is the characteristic function of the cube $[-\alpha, \alpha]^d$. Let $\varepsilon>0$, and let L be a subspace of $L_p(R^d)$. We define

$$K_\varepsilon(\alpha, L, L_p(R^d)) := \min\{n \in Z_+ \mid d_n(P_\alpha(L \cap BL_p(R^d)), L_p(R^d)) < \varepsilon\},$$

where $d_n(A, X)$ denotes the Komolgorov n-width of A in X. The average dimension of L in $L_p(R^d)$ is defined to be

$$\overline{\dim}(L, L_p(R^d)) := \lim_{\varepsilon \to 0} \liminf_{\alpha \to \infty} \frac{K_\varepsilon(\alpha, L, L_p(R^d))}{(2\alpha)^d}.$$

Let $\sigma>0$ and let C be a centrally symmetric subset of $L_p(R^d)$. The average Kolmogorov σ-width (average σ–K width) of C in $L_p(R^d)$ is defined to be

$$\bar{d}_\sigma(C, L_p(R^d)) := \inf_{L} \sup_{x(\cdot) \in C} \inf_{y(\cdot) \in L} \|x(\cdot) - y(\cdot)\|_{L_p(R^d)},$$

where the infimum is taken over all subspaces $L \subset L_p(R^d)$ such that $\overline{\dim}(L, L_p(R^d)) \leqslant \sigma$.

The average linear σ-width (shortly, average σ–L width) of C in $L_p(R^d)$ is defined to be

$$\bar{d}'_\sigma(C, L_p(R^d)) := \inf_{(Y, \Lambda)} \sup_{x(\cdot) \in C} \|x(\cdot) - \Lambda x(\cdot)\|_p,$$

where the infimum is taken over all pairs (Y, Λ) such that, for each pair (Y, Λ), Y is a normed space which is continuously imbedded in $L_p(R^d)$, $C \subset Y$, Λ is a continuous linear operator from Y to $L_p(R^d)$, and $\overline{\dim}(\operatorname{Im} \Lambda, L_p(R^d)) \leqslant \sigma$, while $\operatorname{Im} \Lambda$ denotes the range of the operator Λ.

It follows at once from the definitions that

$$\bar{d}_\sigma(C, L_p(R^d)) \leqslant \bar{d}'_\sigma(C, L_p(R^d)). \tag{1.0}$$

For $\sigma>0$, let Θ_σ be the set of all sequences $\xi = \{\xi_\nu\}_{\nu \in Z^d}$ of points ξ_ν in R^d, $\nu \in Z^d$, satisfying the following conditions:

(i) $|\xi_\nu| \leqslant |\xi_{\nu'}|$, if and only if $|\nu| \leqslant |\nu'|$ for $\nu, \nu' \in Z^d$,

(ii) $\xi_\nu \neq \xi_{\nu'}$, if and only if $\nu \neq \nu'$ for $\nu, \nu' \in Z^d$,

(iii) $$\overline{\text{card}}\, \xi := \liminf_{c \to \infty} \frac{\text{card}(\xi \cap [-c, c]^d)}{(2c)^d} \leqslant \sigma.$$

Here $|\cdot|$ is the usual Euclidean norm, and $\text{card}(\xi \cap [-c, c]^d)$, for any $c > 0$, denotes the number of elements of the set $\xi \cap [-c, c]^d$.

Let $X(R^d)$ be a normed space of functions on R^d with the norm $\|\cdot\|_X$. Put

$$E(A, B, X) := \sup_{x(\cdot) \in A} \inf_{y(\cdot) \in B} \|x(\cdot) - y(\cdot)\|_X$$

for the subsets A, B of $X(R^d)$.

Let $K \subset X(R^d)$. The quantity

$$d(K) := \sup_{x(\cdot),\, y(\cdot) \in K} \|x(\cdot) - y(\cdot)\|_X$$

is called the diameter of K. For $\xi \in \Theta_\sigma$, the information of $f \in K$ is defined by $I_\xi f = \{f(\xi_\nu)\}_{\nu \in Z^d}$. I_ξ is called a standard sampling operator of the average cardinality $\leqslant \sigma$. The quantity

$$\Delta_\sigma(K, X) := \inf_{\xi \in \Theta_\sigma} \sup_{f \in K} d(I_\xi^{-1} I_\xi f \cap K)$$

is called the net width or the minimum information diameter of the set K in the space $X(R^d)$. If K is a balanced and convex subset of $X(R^d)$, then

$$\Delta_\sigma(K, X) = 2 \inf_{\xi \in \Theta_\sigma} \sup\{\|f\|_X \colon I_\xi f = 0, f \in K\}.$$

For any $\xi \in \Theta_\sigma$, a mapping $\varphi\colon I_\xi(K) \to X(R^d)$ is called an algorithm and $\varphi \cdot I_\xi f$ is called a recoverying function of f in $X(R^d)$. Denote by Φ_ξ the set of all algorithms on K. If φ can be extended into a linear operator on the linearized set of K, we call the algorithm φ to be linear. Denote by Φ_ξ^{L} the set of all linear algorithms on the linearized set of K. The quantity

$$E_\sigma(K, X) := \inf_{\xi \in \Theta_\sigma} \inf_{\varphi \in \Phi_\xi} \sup_{f \in K} \|f - \varphi(I_\xi f)\|_X \tag{1.1}$$

is called the minimum intrinsic error of the optimal recovery of the set K in the space X. Taking Φ_ξ^{L} in the place of Φ_ξ in right side of (1.1), we denote the obtained quantity by $E_\sigma^{\mathrm{L}}(K, X)$ and call it the minimum linear intrinsic error. If K is a convex and centrally symmetric subset of X, then by [13], there hold the inequalities

$$\tfrac{1}{2}\,\Delta_\sigma(K, X) \leqslant E_\sigma(K, X) \leqslant E_\sigma^L(K, X). \tag{1.2}$$

1.2. *Besov Classes*

Suppose that $k \in N$. For each $f \in L_p(R^d)$,

$$\Delta_t^k f(x) = \sum_{l=0}^{k} (-1)^{l+k} \binom{k}{l} f(x+lt) \tag{1.3}$$

is the kth difference of the function f at the point x with step t. When $t = (0, ..., 0, t_j, 0, ..., 0)$, we denote $\Delta_t^k f(x)$ by $\Delta_{t_j}^k f(x)$.

Definition 1. Let $k \in N$, $r > 0$, $k - r > 0$, $1 \leqslant \theta \leqslant \infty$, and $1 \leqslant p < \infty$. We say $f \in B_{p\theta}^r(R^d)$ if the f satisfies the following conditions:

(i) $f \in L_p(R^d)$,

(ii) $$\|f\|_{b_{p\theta}^r(R^d)} := \begin{cases} \left\{ \int_{R^d} \left(\dfrac{\|\Delta_t^k f(\cdot)\|_p}{|t|^r} \right)^\theta \dfrac{dt}{|t|^d} \right\}^{1/\theta} < \infty, & 1 \leqslant \theta < \infty, \\ \sup\limits_{|t| \neq 0} \dfrac{\|\Delta_t^k f(\cdot)\|_p}{|t|^r} < \infty, & \theta = \infty. \end{cases} \tag{1.4}$$

By [4], the linear space $B_{p\theta}^r(R^d)$ is a Banach space with the norm

$$\|f\|_{B_{p\theta}^r(R^d)} := \|f\|_p + \|f\|_{b_{p\theta}^r(R^d)}$$

and is called an isotropic Besov space.

Definition 2. Let $\mathbf{k} = (k_1, ..., k_d) \in Z_+^d$, $\mathbf{r} = (r_1, ..., r_d)$, $r_j > 0$, $k_j > r_j$, $j = 1, ..., d$, $1 \leqslant \theta \leqslant \infty$, and $1 \leqslant p < \infty$. We say $f \in B_{p\theta}^{\mathbf{r}}(R^d)$ if it satisfies the following conditions.

(i) $f \in L_p(R^d)$,

(ii) $$\|f\|_{b_{x_j p\theta}^{r_j}(R^d)} := \begin{cases} \left\{ \int_{R^d} \left(\dfrac{\|\Delta_{t_j}^{k_j} f(\cdot)\|_p}{|t_j|^{r_j}} \right)^\theta \dfrac{dt_j}{|t_j|} \right\}^{1/\theta} < \infty, & 1 \leqslant \theta < \infty, \\ \sup\limits_{t_j \neq 0} \dfrac{\|\Delta_{t_j}^{k_j} f(\cdot)\|_p}{|t_j|^{r_j}} < \infty, & \theta = \infty, \end{cases}$$

for $j = 1, ..., d$.

By [4], the linear space $B^{\mathbf{r}}_{p\theta}(R^d)$ is a Banach space with the norm

$$\|f\|_{B^{\mathbf{r}}_{p\theta}(R^d)} := \|f\|_p + \sum_{j=1}^{d} \|f\|_{b^{r_j}_{x_j\, p\theta}(R^d)},$$

and is called an anisotropic Besov space. When $\theta = \infty$, $B^r_{p\theta}(R^d)$ and $B^{\mathbf{r}}_{p\theta}(R^d)$ coincide with the Hölder–Nikolskii spaces $H^r_p(R^d)$ and $H^{\mathbf{r}}_p(R^d)$, respectively. When $r_1 = \cdots = r_d$, from [4], we see that $B^r_{p\theta}(R^d) = B^{r, \ldots, r}_{p\theta}(R^d)$.

For a real vector $\mathbf{M} = (M_1, \ldots, M_d)$, $M_j > 0$, $j = 1, \ldots, d$, we define

$$S^r_{p\theta} b(R^d) := \{ f \in L_p(R^d) : \|f\|_{b^r_{p\theta}(R^d)} \leqslant 1 \},$$

$$S^r_{p\theta} B(R^d) := \{ f \in L_p(R^d) : \|f\|_{B^r_{p\theta}(R^d)} \leqslant 1 \},$$

$$S^{\mathbf{r}}_{p\theta} b(R^d) := \{ f \in L_p(R^d) : \|f\|_{b^{r_j}_{x_j\, p\theta}(R^d)} \leqslant M_j,\ j = 1, \ldots, d \},$$

$$S^{\mathbf{r}}_{p\theta} B(R^d) := \{ f \in L_p(R^d) : \|f\|_{B^{\mathbf{r}}_{p\theta}(R^d)} \leqslant 1 \}.$$

1.3. *Main Results*

Let $\rho > 0$, $v = (v_1, \ldots, v_d)$, $v_i > 0$, $i = 1, \ldots, d$, and $1 \leqslant p < \infty$. Denote by $B^p_v(R^d)$ the set of all those functions from $L_p(R^d)$ in which for each function f the support of the Fourier transform $\hat{f}$ in the distributional sense of f is contained in $[-v_1, v_1] \times \cdots \times [-v_d, v_d]$. The Schwartz theorem states that $B^p_v(R^d)$ coincides with the set of all theose functions from $L_p(R^d)$ which can be continued analytically to entire functions of type $w \leqslant v$. Here $w \leqslant v$ means that $w_j \leqslant v_j$, $j = 1, \ldots, d$, for each $w \in R^d_+ = \{ x \in R^d : x_j > 0, j = 1, \ldots, d \}$.

The approximation features of periodic Besov classes have been studied by many authors [9, 12]. In this paper, we study the problems of the average Kolmogorov width, average linear width, and optimal recovery of the Besov classes $S^r_{p\theta} b(R^d)$, $S^r_{p\theta} B(R^d)$, $S^{\mathbf{r}}_{p\theta} b(R^d)$, and $S^{\mathbf{r}}_{p\theta} B(R^d)$, and obtain the following results.

THEOREM 1. *Let* $\mathbf{k} = (k_1, \ldots, k_d) \in Z^d_+$, $\mathbf{r} = (r_1, \ldots, r_d)$, $k_j > r_j > 0$, $j = 1, \ldots, d$, $1 \leqslant \theta \leqslant \infty$, $1 \leqslant p < \infty$, *and* $\sigma \geqslant 1$. *Then*

(1)
$$\mu \sigma^{-a} \ll \bar{d}_\sigma(\mathscr{A}, L_p(R^d)) \leqslant \bar{d}'_\sigma(\mathscr{A}, L_p(R^d)) \leqslant \sup_{f \in \mathscr{A}} \|f - T_{\rho_1, \ldots, \rho_d} f\|_p \ll \mu \sigma^{-a},$$

where $\mathscr{A} = S^{\mathbf{r}}_{p\theta} b(R^d)$ *or* $S^{\mathbf{r}}_{p\theta} B(R^d)$, $a = (\sum_{j=1}^d 1/r_j)^{-1}$, $\mu = \Pi_{j=1}^d M_j^{a/r_j}$ (*when* $\mathscr{A} = S^{\mathbf{r}}_{p\theta} B(R^d)$, $\mu = 1$), *and* $T_{\rho_1, \ldots, \rho_d} f$ *will be defined in Section* 2.

(2) $B^p_{\boldsymbol{\rho}(\boldsymbol{\sigma})}(R^d)$ *is a weakly asymptotic optimal subspace of average dimension* $\leqslant \sigma$ *for* $\bar{d}_\sigma(\mathscr{A}, L_p(R^d))$, *where* $\rho(\sigma) = (\rho_1(\sigma), \ldots, \rho_d(\sigma))$, $\rho_i(\sigma) > 0$,

is defined by $\rho_j(\sigma) = (\mu^{-1} M_j \sigma^a)^{1/r_j}$ *(when* $\mathscr{A} = S^{\mathbf{r}}_{p\theta} B(R^d)$, $\rho_j(\sigma) = \sigma^{a/r_j}$*),* $j = 1, ..., d$.

Since $B^r_{p\theta}(R^d) = B^{r, ..., r}_{p\theta}(R^d)$, then, taking $M_j = 1$, $r_j = r$, $j = 1, ..., d$, in Theorem 1, we have a corollary of Theorem 1 as follows.

COROLLARY 1. *Let* $k \in N$, $r > 0$, $k - r > 0$, $1 \leqslant \theta \leqslant \infty$, $1 \leqslant p < \infty$, *and* $\sigma \geqslant 1$. *Then*

(1)
$$\sigma^{-r/d} \ll \bar{d}_\sigma(\mathscr{M}, L_p(R^d)) \leqslant \bar{d}'_\sigma(\mathscr{M}, L_p(R^d)) \leqslant \sup_{f \in \mathscr{M}} \| f - T_{\rho_1, ..., \rho_d} f \|_p \ll \sigma^{-r/d},$$

where $\mathscr{M} = S^r_{p\theta} b(R^d)$ *or* $S^r_{p\theta} B(R^d)$.

(2) $B^p_{\rho(\sigma)}(R^d)$ *is a weakly asymptotic optimal subspace of average dimension* σ *for* $\bar{d}_\sigma(\mathscr{M}, L_p(R^d))$, *where* $\rho(\sigma) \geqslant 0$ *is defined by* $\rho(\sigma) = \sigma^{1/d}$.

THEOREM 2. *Let* $k \in N, r > d/p$, $k - r > 0$, $1 \leqslant \theta \leqslant \infty$, $1 \leqslant p < \infty$, *and* $\sigma \geqslant 1$. *Then*

$$\sigma^{-r/d} \ll \tfrac{1}{2} \Delta_\sigma(S^r_{p\theta} B(R^d), L_p(R^d)) \leqslant E_\sigma(S^r_{p\theta} B(R^d), L_p(R^d)) \leqslant E^L_\sigma(S^r_{p\theta} B(R^d), L_p(R^d)) \ll \sigma^{-r/d}.$$

In the remainder of the paper, the constants $c_1, c'_1, c_2, ...$ are dependent only on d, p, θ, and r.

2. PROOF OF THEOREM 1

To prove Theorem 1, we first give some lemmas as follows.

LEMMA 1 (cf. [1]). *Let* $\rho > 0$, $v = (v_1, ..., v_d)$, $v_j > 0$, $j = 1, ..., d$, *and* $1 \leqslant p \leqslant \infty$. *Then*

$$\overline{\dim}(B^p_v(R^d), L_p(R^d)) = \frac{v_1 \cdots v_d}{(\pi)^d}.$$

Let $B(l^N_p)$ denote the unit ball of the space l^N_p.

LEMMA 2 (cf. [11]). *If* $1 \leqslant p \leqslant \infty$, $1 \leqslant n < N$, *then*

$$d_n(B(l^N_p), l^N_p) = 1,$$

where $d_n(A, X)$ denotes the usual Kolmogorov n-width of A in X, while X is a normed linear space and A one of its subsets.

Proof of Theorem 1. To find the upper bound, we first construct the following continuous linear operators from $B^{\mathbf{r}}_{p\theta}(R^d)$ to $L_p(R^d)$. For any $f \in L_p(R^d)$, $t \in R^d$, and natural number l, one has the equation

$$(-1)^{l+1} \Delta_t^l f(x) = (-1)^{l+1} \sum_{j=0}^{l} (-1)^{l+j} \binom{l}{j} f(x+jt)$$

$$= \sum_{j=1}^{l} d_j f(x+jt) - f(x),$$

where $\sum_{j=1}^{l} d_j = 1$. For any real number $v > 0$, let

$$g_v(t) = \lambda_{v,s}^{-1} \left(\frac{\sin vt}{t}\right)^{2s} \qquad (t \in R,\ 2s > 1),$$

be an even entire function of one variable of exponential type $2sv$, where $\lambda_{v,s} = \int_R (\sin vt/t)^{2s}\, dt \asymp v^{2s-1}$, $v \to \infty$. Let $\rho = (\rho_1, ..., \rho_d)$, $\rho_i > 0$, $i = 1, ..., d$. For any $f \in B^{\mathbf{r}}_{p\theta}(R^d)$, set

$$T_{\rho_i}(f, x) := \int_R g_{\rho_i}(t_i)((-1)^{k_i+1} \Delta_{t_i}^{k_i} f(x) + f(x))\, dt_i$$

$$= \int_R g_{\rho_i}(t_i) \sum_{j=1}^{k_i} d_j f(x_1, ..., x_{i-1}, x_i + jt_i, x_{i+1}, ..., x_d)\, dt_i$$

$$= \int_R G_{\rho_i}(t_i - x_i) f(x_1, ..., x_{i-1}, t_i, x_{i+1}, ..., x_d)\, dt_i,$$

where $G_{\rho_i}(t) = \sum_{j=1}^{k_i} (d_j/j)\, g_{\rho_i}(t/j)$. By [4], $G_{\rho_i}(t)$ is an entire function of one variable of exponential type $2\rho_i s$. Set

$$T_{\rho_1 ..., \rho_n}(f, x) := \int_{R^n} G_{\rho_1}(u_1) \cdots G_{\rho_n}(u_n)$$

$$\times f(x_1 + u_1, ..., x_n + u_n, x_{n+1}, ..., x_d)\, du,$$

$1 \leqslant n \leqslant d$. Then $T_{\rho_1, ..., \rho_d}$ is an entire function of d variables of exponential type $\rho = (2s\rho_1, ..., 2s\rho_d)$ (cf. [4]). Let $2s > d + \max\{r_i, i = 1, ..., d\}$. By the Minkowskii and Hölder integral inequality, we have

$$\|f(x)-T_{\rho_1}(f,x)\|_p=\left(\int_{R^d}\left|\int_R g_{\rho_1}(t_1)\,\Delta_{t_1}^{k_1} f(x)\,dt_1\right|^p dx\right)^{1/p}$$

$$\leqslant\int_R \|\Delta_{t_1}^{k_1} f(\cdot)\|_p\, g_{\rho_1}(t_1)\,dt_1$$

$$=\int_R\left(\frac{\|\Delta_{t_1}^{k_1} f(\cdot)\|_p}{|t_1|^{r_1+(1/\theta)}}\right)|t_1|^{r_1+(1/\theta)}\,g_{\rho_1}(t_1)\,dt_1$$

$$\leqslant\left(\int_R\left(\frac{\|\Delta_{t_1}^{k_1} f(\cdot)\|_p}{|t_1|^{r_1+(1/\theta)}}\right)^\theta dt_1\right)^{1/\theta}$$

$$\times\left(\int_R |t_1|^{(r_1+(1/\theta))\,\theta'}\,|g_{\rho_1}(t_1)|^{\theta'}\,dt_1\right)^{1/\theta'}$$

$$\leqslant c_1\,\rho_1^{-r_1}\,\|f\|_{b^{r_1}_{x_1\,p\theta}(R^d)}, \tag{2.1}$$

where $1/\theta+1/\theta'=1$. Moreover, we have

$$\|T_{\rho_1}(f,x)-T_{\rho_1,\rho_2}(f,x)\|_p$$

$$=\Bigg\|\int_R G_{\rho_1}(t_1)\,f(x_1+t_1,x_2,...,x_d)\,dt_1$$

$$-\int_{R^2} G_{\rho_1}(t_1)\,G_{\rho_2}(t_2)\,f(x_1+t_1,x_2+t_2,x_3,...,x_d)\,dt_1\,dt_2\Bigg\|_p$$

$$=\left\|\int_R G_{\rho_1}(t_1)\,h_1(x_1+t_1,x_2,...,x_d)\,dt_1\right\|_p$$

$$\leqslant\int_R g_{\rho_1}(t_1)\,\|h(\cdot)\|_p\,dt_1=\|h(\cdot)\|_p,$$

where $h(x_1,x_2,...,x_d)=f(x_1,x_2,...,x_d)-\int_R G_{\rho_2}(t_2)\,f(x_1,x_2+t_2,x_3,...,x_d)\,dt_2$.

Similarly to (2.1), we can get

$$\|h(\cdot)\|_p\leqslant c_2\,\rho_2^{-r_2}\,\|f\|_{b^{r_2}_{x_2\,p\theta}(R^d)}.$$

Inductively, for $2\leqslant j\leqslant d$, we have

$$\|T_{\rho_1,...,\rho_{j-1}}(f,x)-T_{\rho_1,...,\rho_j}(f,x)\|_p\leqslant c_j\,\rho_j^{-r_j}\,\|f\|_{b^{r_j}_{x_j p\theta}(R^d)}. \tag{2.2}$$

Hence, by (2.2), we have

$$\|f(x)-T_{\rho_1,\dots,\rho_d}(f,x)\|_p$$

$$=\|f(x)-T_{\rho_1}(f,x)+T_{\rho_1}(f,x)-T_{\rho_1,\rho_2}(f,x)+\cdots-T_{\rho_1,\dots,\rho_d}(f,x)\|_p$$

$$\leqslant c\sum_{j=1}^{d}\rho_j^{-r_j}\|f\|_{b^{r_j}_{x_j p\theta}(R^d)}. \tag{2.3}$$

By (2.3), it is easy to see that

$$\|T_{\rho_1,\dots,\rho_d}(f,x)\|_p\leqslant\left(\|f\|_p+\sum_{j=1}^{d}\|f\|_{b^{r_j}_{x_j p\theta}(R^d)}\right)\max\{1,\bar{c}\},$$

where $\bar{c}=c\max\{\rho_j^{-r_j}, 1\leqslant j\leqslant d\}$. Thus, the operator Λ_2: $B^{\mathbf{r}}_{p\theta}(R^d)\to L_p(R^d)$, $\Lambda_2 f(\cdot)=T_{\rho_1,\dots,\rho_d}(f,\cdot)$, is continuous and linear. Put $2s\rho_j=\rho_j(\sigma)=(\mu^{-1}M_j\sigma^a)^{1/r_j}$ (when $\mathscr{A}=S^{\mathbf{r}}_{p\theta}B(R^d)$, put $\rho_j(\sigma)=\sigma^{a/r_j}$). Then, by (2.3) and Lemma 1, we have

$$\bar{d}'_\sigma(\mathscr{A},L_p(R^d))\leqslant\sup_{f\in\mathscr{A}}\|f(x)-T_{\rho_1,\dots,\rho_d}(f,x)\|_p$$
$$\leqslant c\sup_{f\in\mathscr{A}}\sum_{j=1}^{d}\rho_j^{-r_j}\|f\|_{b^{r_j}_{x_j p\theta}(R^d)}\ll\mu\sigma^{-a}.$$

To find the lower bound, let $\lambda=(\lambda_1,\dots,\lambda_d)$, $\lambda_i=(M_i\mu^{-1}(2\sigma)^a)^{-1/r_i}$ (when $\mathscr{A}=S^{\mathbf{r}}_{p\theta}B(R^d)$, $\lambda_i=(2\sigma)^{-a/r_i}$), $i=1,\dots,d$, and the non-zero function $\phi(x)\in C^\infty(R)$ with $\operatorname{supp}(\phi)\subset[0,1]$. For each $j=(j_1,\dots,j_d)\in Z^d$ and any $t=(t_1,\dots,t_d)\in R^d$, set

$$\Phi_{j,\lambda}(t):=\prod_{k=1}^{d}\phi(\lambda_k^{-1}t_k-j_k).$$

Then $\Phi_{j,\lambda}\in C^\infty(R^d)$, and $\operatorname{supp}\Phi_{j,\lambda}\subset\Delta_{j,\lambda}:=[j_1\lambda_1,(j_1+1)\lambda_1]\times\cdots\times[j_d\lambda_d,(j_d+1)\lambda_d]$.

For any $N>0$, set $m_i(N):=[N\lambda_i^{-1}]$. We define a set of functions as follows,

$$L_{m,\lambda}=\operatorname{span}\{\Phi_{j,\lambda}(t)\colon -m_k\leqslant j_k\leqslant m_k-1, k=1,\dots,d\};$$

then, the space $L_{m,\lambda}$ is of dimension $\bar{m}_2=\prod_{i=1}^{d}(2m_i)$. For any $f\in L_{m,\lambda}$, it is easy to see that

$$\operatorname{supp} f\subset[-m_1\lambda_1,m_1\lambda_1]\times\cdots\times[-m_d\lambda_d,m_d\lambda_d]\subset[-N,N]^d.$$

If we write

$$f(t)=\sum_{j_1=-m_1}^{m_1-1}\cdots\sum_{j_d=-m_d}^{m_d-1} a_{j_1,\,...,\,j_d}\Phi_{j,\,\lambda}(t),$$

then

$$\|f\|_p=\left(\prod_{j=1}^{d}\lambda_j\right)^{1/p}\|\phi\|^{d}_{L_p[0,\,1]}\,\|a_j\|_{l_p^{\bar{m}_2}}. \tag{2.4}$$

By the Minkowskii integral inequality, we have

$$\begin{aligned}
\|\Delta_{t_i}^{k_i} f(\cdot)\|_p &= \Bigg\|\int_0^{t_i} du_1\cdots\int_0^{t_i}\frac{\partial^{k_i}}{\partial x_i^{k_i}}\\
&\qquad \times f(x_1,\,...,\,x_i+u_1+\cdots+u_{k_i},\,x_{i+1},\,...,\,x_d)\,du_k\Bigg\|_p\\
&=\Bigg\|\int_0^{t_i} du_1\cdots\int_0^{t_i}\sum_{j_1=-m_1}^{m_1-1}\cdots\sum_{j_d=-m_d}^{m_d-1} a_{j_1,\,...,\,j_d}\,\phi^{(k_i)}\\
&\qquad \times(\lambda_i^{-1}(x_i+u_1+\cdots+u_{k_i})-j_i)\,\lambda_i^{-k_i}\prod_{s\neq i}^{d}\phi(\lambda_s^{-1}x_s-j_s)\,du_{k_i}\Bigg\|_p\\
&\leqslant\int_0^{|t_i|}\cdots\int_0^{|t_i|}\Bigg\|\sum_{j_1=-m_1}^{m_1-1}\cdots\sum_{j_d=-m_d}^{m_d-1} a_{j_1,\,...,\,j_d}\phi^{(k_i)}\\
&\qquad \times(\lambda_i^{-1}(x_i+u_1+\cdots+u_{k_i})-j_i)\,\lambda_i^{-k_i}\prod_{s\neq i}^{d}\phi(\lambda_s^{1}x_s-j_s)\Bigg\|_p\,du\\
&=\left(\prod_{j=1}^{d}\lambda_j\right)^{1/p}\lambda_i^{-k_i}\,\|\phi^{(k_i)}\|_{L_p[0,\,1]}\,\|\phi\|^{d-1}_{L_p[0,\,1]}\,\|a_j\|_{l_p^{\bar{m}_2}}\,|t_i|^{k_i}\\
&=c_{i_1}\left(\prod_{j=1}^{d}\lambda_j\right)^{1/p}\lambda_i^{-k_i}\,|t_i|^{k_i}\,\|a_j\|_{l_p^{\bar{m}_2}}.
\end{aligned} \tag{2.5}$$

By (2.4), we have

$$\|\Delta_{t_i}^{k_i} f(\cdot)\|_p\leqslant c_{i_2}\|f\|_p\leqslant c_{i_3}\left(\prod_{j=1}^{d}\lambda_j\right)^{1/p}\|a_j\|_{l_P^{\bar{m}_2}}. \tag{2.6}$$

Thus, by (2.5) and (2.6), we have

$$\|\Delta_{t_i}^{k_i} f(\cdot)\|_p\leqslant c_{i_4}\left(\prod_{j=1}^{d}\lambda_j\right)^{1/p}\|a_j\|_{l_p^{\bar{m}_2}}\min\{1,\,(\lambda_i^{-1}\,|t_i|)^{k}\}. \tag{2.7}$$

Further, for $1 \leqslant \theta < \infty$, we get

$$\|f\|_{b^{r_i}_{t_i\, p\theta}(R^d)} = \left(\int_R \left(\frac{\|\Delta^{k_i}_{t_i} f(\cdot)\|_p}{|t_i|^{r_i}} \right)^{\theta} \frac{dt_i}{|t_i|} \right)^{1/\theta}$$

$$\leqslant c_{i_5} \left(\prod_{j=1}^{d} \lambda_j \right)^{1/p} \|a_j\|_{l_p^{\bar{m}_2}}$$

$$\times \left(\int_0^{\lambda_i} \lambda_i^{-k_i\theta} R^{(k_i - r_i)\theta - 1}\, dR + \int_{\lambda_i}^{\infty} R^{-r_i\theta - 1}\, dR \right)^{1/\theta}$$

$$= c_i \left(\prod_{j=1}^{d} \lambda_j \right)^{1/p} \lambda_i^{-r_i} \|a_j\|_{l_p^{\bar{m}_2}}. \tag{2.8}$$

For the case $\theta = \infty$, (2.8) is also valid. Set

$$\delta_N := \left(\prod_{j=1}^{d} \lambda_j \right)^{1/p} \mu^{-1}(2\sigma)^a c_N \quad (c_N = \|\phi\|^d_{L_p[0,1]} + \max\{c_i : i = 1, ..., d\}),$$

$$Q_N(\delta_N) := \{ f \in L_{m,\lambda} : \|a_j\|_{l_p^{\bar{m}_2}} \leqslant \delta_N^{-1} \}. \tag{2.9}$$

Then it is easy to see that $Q_N \subset \mathscr{A}$.

Now, we estimate the quantity $\bar{d}_\sigma(\mathscr{A}, L_p(R^d))$ from below, for $1 \leqslant p < \infty$. Let A be a subspace of $L_p(R^d)$ with the average dimension $\overline{\dim}(A, L_p(R^d)) \leqslant \sigma$. By the definition of the average dimension, for any $N > 0$, and $\varepsilon > 0$, there exists a subspace $A_1 \subset L_p(I_N^d)$ with the dimension $K := \dim A_1 = K_\varepsilon(N, A, L_p(I_N^d))$ such that

$$E(B(A)|_{I_N^d}, A_1, L_p(I_N^d)) \leqslant \varepsilon,$$

where $B(A)$ denotes the unit ball of the space A. Moreover, for any $g \in A$, we have

$$e(g|_{I_N^d}, A_1, L_p(I_N^d)) \leqslant \varepsilon \|g\|_p.$$

Here we have put $e(x, B, X) := \inf_{y(\cdot) \in B} \|x(\cdot) - y(\cdot)\|_X$, for any element x of the subset B of the linear normed space X. Thus, for any $f \in \mathscr{A}$ and any $g \in A$, we have

$$\|f - g\|_p \geqslant \|f - g\|_{L_p(I_N^d)}$$

$$\geqslant e(f, A_1, L_p(I_N^d)) - e(g, A_1, L_p(I_N^d))$$

$$\geqslant e(f, A_1, L_p(I_N^d)) - \varepsilon \|g\|_p$$

$$\geqslant e(f, A_1, L_p(I_N^d)) - \varepsilon \|f - g\|_p - \varepsilon \|f\|_p.$$

This is

$$(1+\varepsilon)\,\|f-g\|_p \geqslant e(f, A_1, L_p(I_N^d)) - \varepsilon\,\|f\|_p.$$

Further, we have

$$(1+\varepsilon)\,E(\mathscr{A}, A, L_p(R^d)) \geqslant E(Q_N, A_1, L_p(I_N^d)) - \varepsilon \sup_{f \in Q_N} \|f\|_p. \tag{2.10}$$

By (2.4), (2.9), and Lemma 2, we have

$$\begin{aligned} E(Q_N, A_1, L_p(R^d)) &\geqslant c' \left(\prod_{j=1}^{d} \lambda_j\right)^{1/p} \delta_N^{-1} d_K(B(l_p^{\bar{m}_2}), l_p^{\bar{m}_2}) \\ &= c' \left(\prod_{j=1}^{d} \lambda_j\right)^{1/p} \delta_N^{-1} = c\mu\sigma^{-a}. \end{aligned} \tag{2.11}$$

By (2.10) and (2.11), then by letting $N \to \infty$ and $\varepsilon \to 0$, we get

$$\bar{d}_\sigma(\mathscr{A}, L_p(R^d)) \gg \mu\sigma^{-a}.$$

By (1.0), we complete the proof of Theorem 1.

3. PROOF OF THEOREM 2

Let l be an even number, $0 < \alpha < l$, as in [10], for any $f \in L_p(R^d)$, we define the differential operator D^α by the following

$$(D^\alpha f)(x) := \lim_{\substack{L_p(R^d) \\ \varepsilon \to 0^+}} (D_\varepsilon^\alpha f)(x). \tag{3.1}$$

Here D_ε^α is an operator defined by

$$(D_\varepsilon^\alpha f)(x) := \frac{1}{m_{d,l}(\alpha)} \int_{|y| \geqslant \varepsilon} \frac{\Delta_y^l f(x)}{|y|^{d+\alpha}}\, dy,$$

$$m_{d,l}(\alpha) := \int_{R^d} \frac{(e^{iy_1/2} - e^{-iy_1/2})^l}{|y|^{d+\alpha}}\, dy,$$

where $y = (y_1, y_2, ..., y_d) \in R^d$. For $\rho > 0$, set

$$S_{\beta,\rho} f(x) := \sum_{v \in Z^d} f\left(\frac{v}{\rho}\right) L_\beta(\rho x - v),$$

where $L_\beta(x)$ satisfies $L_\beta(v)=\delta_{v,0}$, $v\in Z^d$, and is defined by its generalized Fourier transform

$$\widehat{L_\beta}(y)=(2\pi)^{-d/2}\frac{|y|^{-\beta}}{\sum_{v\in Z^d}|y-2v\pi|^{-\beta}}.$$

For $\alpha>0$, put

$$W_p^\alpha(R^d):=\{f\in L_p(R^d)\cap C(R^d):\|D^\alpha f\|_p<\infty\}.$$

By [5], we can get

LEMMA 3. *Let* $\alpha>d/p$, $1\leqslant p<\infty$, $\rho>0$, $\beta\geqslant\alpha$, *and* $\beta>d$. *Then, for all* $f\in W_p^\alpha(R^d)(\alpha-d\neq 2,4,...)$, *there exists a constant* $c=c(\alpha,p,\beta)>0$, *such that*

$$\|f-S_{\beta,\rho}f\|_p\leqslant c\rho^{-\alpha}\|D^\alpha f\|_p.$$

For $\lambda>0$, denote by $SB_\lambda^p(R^d)$ the set of all entire functions of spherical exponential type $\leqslant\lambda$ (see [4]), we have the following inequality.

LEMMA 4. *Let* $\lambda>0, 1\leqslant p<\infty$, *and* $\sigma>0$. *Then, for all* $f\in SB_\lambda^p(R^d)$, *there exists a constant* $c=c(\alpha,p)>0$ *such that*

$$\|D^\alpha f\|_p\leqslant c\lambda^\alpha\|f\|_p.$$

Proof. By the definition of $D^\alpha f$ in (3.1), we have

$$\|D^\alpha f\|_p\leqslant c_1\int_{R^d}\frac{\|\Delta_y^l f\|_p}{|y|^{d+\alpha}}dy. \tag{3.2}$$

Since $f\in SB_\lambda^p(R^d)$, it is easy to verify that

$$\|\Delta_y^l f\|_p\leqslant c_2\|f\|_p\min\{1,(|y|\,\lambda)^l\}. \tag{3.3}$$

Hence, by (3.2) and (3.3), we have

$$\|D^\alpha f\|_p\leqslant c_3\|f\|_p\left(\lambda^l\int_0^{\lambda^{-1}}t^{l-\alpha-1}\,dt+\int_{\lambda^{-1}}^\infty t^{-\alpha-1}\,dt\right)=c\lambda^\alpha\|f\|_p.$$

We complete the proof of Lemma 4.

Proof of Theorem 2. Upper estimate. For any $f\in S_{p\theta}^rB(R^d)$, by [4], f may be represented in the form of a series converging to it in the sense of

$L_p(R^d)$: $f(x)=\sum_{l\in Z_+} Q_{a^l}(x)$, $Z_+ =: \{0, 1, ...\}$, whose terms are entire functions of spherical exponential type a^l, $a>1$, such that

$$\|f\|_{B^r_{p\theta}(R^d)} \asymp \begin{cases} \left\{ \sum_{l\in Z_+} a^{lr\theta} \|Q_{a^l}\|_p^\theta \right\}^{1/\theta}, & 1\leqslant \theta<\infty, \\ \sup_{l\in Z_+} a^{lr} \|Q_{a^l}\|_p, & \theta=\infty. \end{cases} \tag{3.4}$$

Let $\alpha\in(d/p, r)$ and $\beta>r$. For $\rho>1$, let N be a natural number and satisfy $\rho<a^N<2\rho$. By Lemma 3 and Lemma 4, we have

$$\begin{aligned} \|Q_{a^l}-S_{\beta,\rho}Q_{a^l}\|_p &\leqslant c_1 \rho^{-\beta} \|D^\beta Q_{a^l}\|_p \\ &\leqslant c\rho^{-\beta} a^{l\beta} \|Q_{a^l}\|_p, \end{aligned} \tag{3.5}$$

for $0\leqslant l\leqslant N-1$, and

$$\|Q_{a^l}-S_{\beta,\rho}Q_{a^l}\|_p \leqslant c_1\, \rho^{-\alpha} \|D^\alpha Q_{a^l}\|_p \leqslant c\rho^{-\alpha} a^{l\alpha} \|Q_{a^l}\|_p, \tag{3.6}$$

for $l\geqslant N$. Thus, by (3.5) and (3.6), we have

$$\begin{aligned} \|f-S_{\beta,\rho}f\|_p &\leqslant \sum_{l=0}^{\infty} \|Q_{a^l}-S_{\beta,\rho}Q_{a^l}\|_p = \left(\sum_{l=0}^{N-1}+\sum_{l=N}^{\infty}\right) \|Q_{a^l}-S_{\beta,\rho}Q_{a^l}\|_p \\ &\ll \rho^{-\beta} \sum_{l=0}^{N-1} a^{l\beta} \|Q_{a^l}\|_p + \rho^{-\alpha} \sum_{l=N}^{\infty} a^{l\alpha} \|Q_{a^l}\|_p. \end{aligned} \tag{3.7}$$

Further, by (3.4) and Hölder inequality, we have

$$\begin{aligned} \sum_{l=0}^{N-1} a^{l\beta} \|Q_{a^l}\|_p &\leqslant \left(\sum_{l=0}^{N-1} a^{lr\theta} \|Q_{a^l}\|_p^\theta\right)^{1/\theta} \left(\sum_{l=0}^{N-1} a^{l(\beta-r)\theta'}\right)^{1/\theta'} \\ &\ll \|f\|_{B^r_{p\theta}(R^d)}\, a^{N(\beta-r)} \ll \rho^{\beta-r} \|f\|_{B^r_{p\theta}(Rd)} \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} \sum_{l=N}^{\infty} a^{l\alpha} \|Q_{a^l}\|_p &\leqslant \left(\sum_{l=N}^{\infty} a^{lr\theta} \|Q_{a^l}\|_p\right)^{1/\theta} \left(\sum_{l=N}^{\infty} a^{l(\alpha-r)\theta'}\right)^{1/\theta'} \\ &\ll \|f\|_{B^r_{p\theta}(R^d)}\, a^{-N(r-\alpha)} \ll \|f\|_{B^r_{p\theta}(R^d)}\, \rho^{\alpha-r}, \end{aligned} \tag{3.9}$$

for $1<\theta<\infty$. By (3.7) to (3.9), we get

$$\begin{aligned} \|f-S_{\beta,\rho}f\|_p &\ll (\rho^{-\beta}\cdot\rho^{\beta-r}+\rho^{-\alpha}\cdot\rho^{\alpha-r}) \|f\|_{B^r_{p\theta}(R^d)} \\ &\ll \rho^{-r} \|f\|_{B^r_{p\theta}(R^d)}. \end{aligned} \tag{3.10}$$

For $\theta=1, \infty$, (3.10) is also valid. Let $\rho=\sigma^{1/d}$. By (3.10), we have

$$E_L^\sigma(S_{p\theta}^r B(R^d), L_p(R^d)) \leqslant \sup_{f \in S_{p\theta}^r B(R^d)} \|f - S_{\beta, \sqrt[d]{\sigma}} f\|_p \ll \sigma^{-r/d}.$$

Lower estimate. For any $\xi \in \Theta_\sigma$, i.e.,

$$\overline{\text{card}}\ \xi = \liminf_{c \to \infty} \frac{\text{card}(\xi \cap [-c, c]^d)}{(2c)^d} \leqslant \sigma,$$

there exists a cube with the form

$$Q=\{x \in R^d : a_j \leqslant x_j \leqslant a_j + m^{-1},\ j=1, ..., d\},$$

$m=(2\sigma)^{1/d}$, such that its interior Int Q does not contain any point of ξ; that is, Int $Q \cap \xi = \varnothing$. This is easy to see from the fact that $|Q|=(2\sigma)^{-1}$. Let the univariate function $\lambda(t)$, $t \in R$, satisfy the following conditions: $\lambda(t) \in C^\infty(R)$, supp $\lambda \subset [0, 1]$, $0 \leqslant \lambda(t) \leqslant 1$ for $t \in R$, and $\lambda(t)=1$ for $t \in [\frac{1}{4}, \frac{3}{4}]$. For $1 \leqslant p < \infty$, set

$$f_0(x)=\eta \prod_{j=1}^d \lambda(m(x_j - a_j)),$$

where η is a positive constant to be determined. It is easy to see that $f_0(x) \in C^\infty(R^d)$, supp $f_0 \subset Q$, $I_\xi f_0 = 0$, and

$$\|f_0\|_p \leqslant \eta m^{-d/p}. \tag{3.11}$$

It is easy to see that

$$\|\Delta_t^k f_0(\cdot)\|_p \leqslant c_1 \eta m^{-d/p} \min\{1, (m\,|t|)^k\}.$$

Further, we get

$$\|f_0\|_{b_{p\theta}^r(R^d)} \leqslant c_2 \eta m^{d/p} \left(\int_0^{m^{-1}} m^{k\theta} t^{(k-r)\theta-1}\, dt + \int_{m^{-1}}^\infty t^{-r\theta-1}\, dt \right)^{1/\theta}$$
$$\leqslant c_3 \eta m^{-d/p+r}. \tag{3.12}$$

For the case $\theta=\infty$, (3.12) is also valid. By (3.11) and (3.12), if we let $\eta = m^{d/p-r} c^{-1}$ ($c=c_3+1$), then $f_0 \in S_{p\theta}^r B(R^d)$. Set

$$\bar{Q}=\left\{x \in R^d : a_i + \frac{1}{4m} \leqslant x_i \leqslant a_i + \frac{3}{4m},\ i=1, ..., d\right\}.$$

For any $\xi \in \Theta_\sigma$, we have

$$d(I_\xi^{-1}(I_\xi f_0) \cap S^r_{p\theta} B(R^d)) \geqslant \|f_0\|_p \geqslant \|f_0\|_{L_p(\bar{Q})} = \eta\, |\bar{Q}|^{1/p}$$
$$\geqslant c m^{d/p - r}(2m)^{-d/p} \gg \sigma^{-r/d}. \qquad (3.13)$$

By (3.13) and the definition of $\Delta_\sigma(S^r_{p\theta} B(R^d), L_p(R^d))$, we get

$$\Delta_\sigma(S^r_{p\theta} B(R^d), L_p(R^d)) \gg \sigma^{-r/d}.$$

By (1.2), we complete the proof of Theorem 2.

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